SOME PROPERTIES OF THE CUMULATIVE RESIDUAL ENTROPY

Osman Kamari
College of Administration and Economics
University of Human Development, **Iraq**
Email: Osman.kamari@uhd.edu.iq

ABSTRACT

The entropy functions are useful tools to measure the uncertainty. Recently, Rao et al (2004), Navarro et al (2010) defined the cumulative residual entropy and dynamic cumulative past entropy, respectively, as some new measures of uncertainty. They studied some properties and applications of these measures. In this paper, we obtain some results connecting these measures. We obtain cumulative residual entropy for random sample and we show how dynamic cumulative past entropy dominates past entropy.

Keywords: Uncertainty, Cumulative residual entropy, Dynamic cumulative past entropy.

INTRODUCTION

Shannon (1948) proposed a measure of uncertainty in a discrete distribution based on the Boltzmann entropy of classical statistical mechanics. He called it the entropy. The Shannon entropy of a discrete distribution F is defined by

$$H(F) = -\sum_{i=1}^{n} p_i \log p_i \,, \qquad (1-1)$$

Where p_i 's are the probabilities computed from the distribution F. However, extension of this notion to continuous distribution poses some challenges. A straightforward extension of the discrete case to continuous distributions F with density f, called differential entropy, reads

$$h(F) = -\int_{-\infty}^{+\infty} f(x) \log f(x)\ dx. \qquad (1-2)$$

This is a measure of the uncertainty of the lifetime of a unit or a system. The idea is that a unit with great uncertainty is less reliable than a unit with low uncertainty. The concept of entropy is important for studies in many areas of engineering such as thermodynamics, mechanics, or digital communications. In Shannon's entropy, discrete values and absolutely continuous distributions are treated in a somewhat different way through entropy and differential entropy, respectively. Recently, Rao et al (2004) defined a new measure of uncertainty, the cumulative residual entropy (CRE) through

$$\mathcal{E}(X) = -\int_{\mathcal{R}_+^N} P(|X| > x) \log P(|X| > x)\ dx, \qquad (1-3)$$

Where $X = (X_1, X_2, \dots, X_N)$, $x = (x_1, x_2, \dots, x_N)$, and $|X| > x$ means $|X_i| > x_i$ and

$$\mathcal{R}_+^N = \{x_i \in \mathcal{R}^N;\ x_i \geq 0\}.$$

This measure is based on the complementary cumulative distribution function (survival function). Clearly, this formula is valid both for a discrete or an absolutely continuous random variable or with both a discrete and an absolutely continuous part, because it resorts to the survival function of $|X|$. In addition, unlike Shannon differential entropy it is always positive, while preserving many interesting properties of Shannon entropy. CRE is defined in the univariate case and for non-negative random variables as follows:

$$\mathcal{E}(X) = CRE(X) = -\int_0^{\infty} \bar{F}(x) \log \bar{F}(x)\, dx, \qquad (1-4)$$

Where $\bar{F}(x) = P(X > x)$ is the survival function.
By application of these entropy functions to the reversed residual lifetime (past inactivity), we obtain dynamic measures of uncertainty which can measure the aging process.
Di Cresceno and Longbardi (2002) introduced the entropy of the past lifetime $_tX = [t - X|X < t]$ as a dynamic measure of uncertainty as follows

$$H(X;[t]) = -\int_0^t \frac{f(x)}{F(t)}\, log\, \frac{f(x)}{F(t)}\, d\,x. \qquad (1-5)$$

In this paper, we obtain some results on these functions, correcting some mistakes in the preceding literature. The paper is organized as follows. In section 2, we study CRE. Section 3 is devoted to study DCPE. Finally conclusion is given in section 4.

ON CUMULATIVE RESIDUAL ENTROPY

As a measure of uncertainty, Rao et al (2004) proposed the cumulative residual entropy(CRE) of X defined by

$$\mathcal{E}(X) = -\int_{\mathcal{R}_+^N} P(|X| > x)\, log\, P(|X| > x)\ dx,$$

Where N is the dimension of the random vector X. Note that CRE is always non-negative and its definition is valid in the continuous and discrete domains or with both a discrete and an absolutely continuous part. Other properties of this measure can be seen in Rao et al (2004) and Rao (2005). Next we give a few examples and some new properties for the CRE.

Example 2.1. Let the random variable X have the density function

$$f(x) = \frac{1}{\sigma} exp\left\{-\frac{(x-\mu)}{\sigma}\right\}, \qquad x \geq \mu\,, \mu, \sigma > 0\,.$$

Then its CRE is computed as follows:

$$\mathcal{E}(X) = \int_{\mu}^{\infty} e^{-\frac{(x-\mu)}{\sigma}} \left(\frac{(x-\mu)}{\sigma}\right) dx = \sigma.$$

Example 2.2. (CRE of the geometric distribution)
The geometric distribution with mean $1/p$ has the probability function

$$f(x) = pq^x, \qquad q = 1-p, \qquad x = 0,1, \ldots$$

The its survival function is follows

$$p(X > x) = q^{x+1}, x = 0,1, \ldots$$

Correspondingly, the CRE of the geometric distribution is

$$CRE(X) = -\sum_{x=0}^{\infty} q^{x+1}\, log\, q^{x+1} =$$

$$-\sum_{x=0}^{\infty} q^{x+1}(x+1)\, log\, q = -\,log\, q\left[\sum_{x=0}^{\infty} x\, q^{x+1} + \sum_{x=0}^{\infty} q^{x+1}\right] = -\frac{q\, log\, q}{(1-q)^2}.$$

The following example shows that example of 3 of Rao et al (2004) is not true, they used the survival function of X to obtain CRE of normal distribution, we should use the survival function of $|X|$ instead.

Example 2.3. (CRE of the normal distribution)
The normal probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

Where μ is the mean and σ^2 is the variance. The cumulative distribution function is

$$F(x) = 1 - erfc(\frac{x-\mu}{\sigma})$$

Where *erfc* is the error function

$$erfc(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt,$$

And the survival function of |X| is

$$p(|X| > x) = 1 + erfc\left(\frac{x-\mu}{\sigma}\right) - erfc\left(\frac{-x-\mu}{\sigma}\right),$$

Then the CRE of the normal distribution is

$$\begin{aligned}\mathcal{E}(X) &= -\int_0^\infty p(|X| > x)\, log\, p(|X| > x)\, dx \\ &= -\int_0^\infty \left(1 + erfc\left(\frac{x-\mu}{\sigma}\right) - erfc\left(\frac{-x-\mu}{\sigma}\right)\right) log(1 + erfc\left(\frac{x-\mu}{\sigma}\right) \\ &\quad - erfc\left(\frac{-x-\mu}{\sigma}\right))\, dx,\end{aligned}$$

Where CRE of X is computed numerically. If X has the standard normal, that is, its mean is 0 and its variance is 1, then

$$\mathcal{E}(X) = -\int_0^\infty P(|X| > x)\, log\, P(|X| > x)\;\, dx = -\int_0^\infty 2\big(1 - F(x)\big)\, log(\,2(1 - F(x)))\;\, dx,$$

CRE is computed in this part numerically, too. It is clear that if x=0 then CRE(X)=0.

For the CRE, we obtain the following results.

Theorem 2.1. If X_i , i =1,2,…,n be independent, then

$$\mathcal{E}(X) = \sum_{i=1}^{n}\left[\left(\prod_{i\neq j} E\big(|X_j|\big)\right)\mathcal{E}(X_i)\right].$$

Proof:

$$\begin{aligned}\mathcal{E}(X) &= -\int_{\mathcal{R}_+^N} P(|X| > x)\, log\, P(|X| > x)\;\, dx \\ &= -\int_{\mathcal{R}_+^N} (\prod_{i=1}^{n} P(|X_i| > x_i)\, log \prod_{i=1}^{n} P(|X_i| > x_i))\, dx \\ &= -\int_{\mathcal{R}_+^N} (\prod_{i=1}^{n} P(|X_i| > x_i) \sum_{i=1}^{n} log\, P(|X_i| > x_i))\;\, dx \\ &= \sum_{i=1}^{n}\left(\prod_{i\neq j} E\big(|X_j|\big)\right)\mathcal{E}(X_i).\end{aligned}$$

Theorem 2.2. Let $X_1, X_2, \dots, X_n$ be a random sample, then

$$\mathcal{E}(X) = n\,\mathcal{E}(X_1)\,(E(|X_1|))^{n-1}.$$

Proof: The proof follows by applying theorem 2.1.

ON DYNAMIC CUMULATIVE PAST ENTROPY

Let the random variable X denote the lifetime (X ≥ 0 with probability one) of a unit, having an absolutely continuous distributions function $F_X(t) = p(X > t)$ and the density function $f_X(t)$. For t>0, let the random variable ${}_t\mathrm{X} = [\mathrm{t} - \mathrm{X}|\mathrm{X} < t]$ denote the time elapsed after failure till time t, given that the unit has already failed by time t. we call the random variable ${}_t$X, the reversed residual life (or inactivity time). To illustrate the importance of the random variables of the form ${}_t$X we give an example here. Let us assume that, at time t, one has undergone a medical test to check for a certain disease. Let us assume that the test is positive. If we denote by X the age when the patient was infected, then it is known that $X < t$. Now the question is, how much time has elapsed since the patient had been infected by this disease.

The survival function of ${}_t$X is given by

$$\bar{F}(x) = P\left({}_tX > x\right) = P(t - X > x|X < t\,) = \frac{F_X(t-x)}{F_X(t)}, \quad 0 < x < t.$$

Various dynamic information measures have been proposed in order to describe the uncertainty in suitable functional of the random lifetime X of a system. particularly, measure of uncertainty in past lifetime distribution plays an important role in the context of Information Theory, Forensic Science and other related fields.

Di Cresceno and Longbardi (2002) considered the Shannon entropy for the reversed residual lifetime as a dynamic measure of uncertainty that is called past entropy and is defined by

$$H(X;[t]) = -\int_0^t \frac{f(x)}{F(t)}\, log\, \frac{f(x)}{F(t)}\, dx. \qquad (3-1)$$

Obviously $H(X;[\infty]) = h(X)$ and $H(X;[t]) \in (-\infty,\infty)$. This function can also be used to describe the ageing process and can be related to other ageing measures such as the reversed failure (or hazard) rate and the mean inactivity time.

Navarro et al (2010) considered the corresponding dynamic measure ,the dynamic cumulative past entropy (DCPE) of X and it is defined as the CRE of ${}_t$X.

$$\bar{\mathcal{E}}(X;t) = -\int_0^t \frac{F_X(x)}{F_X(t)} log \frac{F_X(x)}{F_X(t)} dx, \qquad (3-2)$$

It is clear that $\bar{\mathcal{E}}(X;\infty) = \mathcal{E}(X)$ and it can be easily seen that for each t, $t \geq 0$, $\bar{\mathcal{E}}(X;t)$ possesses all the properties of the CRE (1-4). It worth noting that $\bar{\mathcal{E}}(X;t)$ provides a dynamic information measure for measuring the information of the reversed residual lifetime.

Example 3.1. Let X be distributed uniformly on (0,1). Then

$$\bar{\mathcal{E}}(X;t) = -\int_0^t \frac{x}{t} log \frac{x}{t} dx = \frac{1}{2},$$

Which DCPE for uniform distribution does not depend on t.

Example 3.2. Let X be distributed as beta$(\alpha, 1)$ (power distribution) with probability density function

$$f(x) = \alpha x^{\alpha-1}, \qquad 0 < x < 1, \qquad \alpha > 0.$$

Then it can be easily seen that DCPE of the power distribution is

$$\bar{\mathcal{E}}(X;t) = \frac{\alpha t}{(\alpha+1)^2}.$$

This shows that the DCPE for power distribution is an increasing function of t. Hence as t gets larger the uncertainty gets larger.

In the following theorem, we show that DCPE dominates the past entropy (which may exist when *X* has density)

Theorem 3.1. Let X ≥ 0 have density *f*, then

$$\bar{\mathcal{E}}(X;t) \geq C exp\{H(X;[t])\}. \qquad (3-3)$$

Where

$$C = exp\left\{\int_0^1 log(x|log\, x|)dx\right\} \cong 0.2065$$

Proof: Let $F(x) = P[X > x] = \int_x^\infty f(u)du$, using the log-sum inequality and replacement past entropy, we have

$$\int_0^t \frac{f(x)}{F(t)}\, log\left(\frac{f(x)/F(t)}{\frac{F(x)}{F(t)}\left|log\frac{F(x)}{F(t)}\right|}\right)\, dx \geq log\left(\frac{1}{\bar{\mathcal{E}}\,(X;t)}\right), \qquad (3-4)$$

Finally, a change of variable gives

$$\int_0^t \frac{f(x)}{F(t)} log\left(\frac{F(x)}{F(t)}\left|log\frac{F(x)}{F(t)}\right|\right) dx = \int_0^1 log(x|log\, x|)\, dx. \qquad (3-5)$$

Therefore, we have from (3-4) and (3-5) the following equation:

$$\bar{\mathcal{E}}\,(X;t) \geq exp\left\{\int_0^1 log(x|log\, x|)\, dx + H(X;[t]\right\}, \qquad (3-6)$$

Hence the proof is complete.

CONCLUSION

The entropy functions are quantities which investigate measure of uncertainty in a random variable. By application of these entropy functions to the reversed residual lifetime, we obtain dynamic measures of uncertainty which can measure the aging process. In this paper, we obtained some properties on the cumulative residual entropy and dynamic cumulative past entropy. We obtained CRE of random sample and we showed that DCPE dominates the past entropy**.**

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